SCHEDAE INFORMATICAE
VOLUME 12 2003

Domain Theory as a Tool for Topology – a Case Study

Paweł Waszkiewicz
Institute of Computer Science, Jagiellonian University, Nawojki 11, Kraków, 30-072 Poland, e-mail: pqw@ii.uj.edu.pl

Abstract. In this paper we adopt a certain view on continuous posets and see them as models of their spaces of maximal elements, which are most often topologies rich in structure. Adopting this perspective seems to be fruitful: we are often able to match structural properties of the modelling poset to properties of the modelled space. It was discovered by Mike Reed and Keye Martin two years ago that existence of a measurement on the model corresponds to existence of a development for the modelled topological space. We present an elementary proof of this fact and show how one can use this result to give a new proof to one of the first metrization theorems in Topology.

Keywords: Continuous dcpo, domain, development, metrization.

1. Introduction

Consider the poset $\mathbf{I}\mathbb{R}$ of closed, connected intervals of the real line ordered by inverse inclusion. One readily sees that its subspace of maximal elements is precisely the set of real numbers. Moreover, when equipped with the subspace Scott topology, the maximal elements of $\mathbf{I}\mathbb{R}$ are homeomorphic to $\mathbb{R}$ in the Euclidean topology. Therefore, we can treat the poset (actually: the continuous dcpo) of intervals as a model, or an approximation structure, for real numbers. To be precise, by a model of a topological space we understand a continuous poset and a homeomorphism between the space and the subset of maximal points of the poset equipped with the subspace Scott topology. At this point one can ask two basic questions: Which topological spaces have models? How are structural properties of modelled spaces reflected in the structure of the model? In this case study we present some fundamental relations between the spaces and their models using a recent theory of measurements developed by Keye Martin in [5]. Our main goal is to show that domain-theoretic tools that we develop in this note are indeed valuable for Topology. This thesis can be supported by the fact that we are able to shed new light on some classical theorems. As a supporting example, we propose a new proof of the metrization theorem by Alexandroff and Urysohn.

Our exposition is based on the author's doctoral dissertation [6].

2. Domain theory

We review some basic notions from domain theory, mainly to fix the language and notation. See [1] for more information.

2.1. Posets

Let P be a poset. A subset $A \subseteq P$ of P is *directed* if it is nonempty and any pair of elements of A has an upper bound in A. If a directed set A has a supremum, it is denoted $\bigsqcup^{\uparrow} A$. A poset P in which every directed set has a supremum is called a *dcpo.* The subset of maximal elements of a poset P is denoted as $\max P$. A pair of elements $x, y \in P$ is *consistent* (*bounded*), denoted $x\uparrow y$, if there exists an element $z \in P$ such that $z \sqsupseteq x, y$. The contrary case is written as $x\#y$.

2.2. Approximation

Let x and y be elements of a poset P. We say that x *approximates* (*is way-below*) y if for all directed subsets A of P, $y \sqsubseteq \bigsqcup^{\uparrow} A$ implies $x \sqsubseteq a$ for some $a \in A$. We denote it as $x \ll y$. If $x \ll x$, then x is called a *compact* element. The subset of compact elements of a poset P is denoted $K(P)$.

Now, $\mathord{\downarrow\!\!\downarrow} x$ is the set of all approximants of x below it, i.e. $\{y \in P \mid y \ll x\}$. Similarly, $\mathord{\uparrow\!\!\uparrow} x$ is defined as $\{y \in P \mid x \ll y\}$. We say that a subset B of a poset P is a (*domain-theoretic*) *basis* for P if for every element x of P, the set $\mathord{\downarrow\!\!\downarrow} x \cap B$ is directed with supremum x. A poset is called *continuous* if it has a basis. One can show that a poset P is continuous iff $\mathord{\downarrow\!\!\downarrow} x$ is directed with supremum x, for all $x \in P$. A poset is called a *domain* if it is a continuous dcpo. Note that $K(P) \subseteq B$ for any basis B of P. If a poset admits a countable basis, we say that it is ω-continuous. A poset P is *ideal* if every element x of P is either compact or maximal (or both).

The poset $[0, \infty)^{op}$ is a domain without a least element. We use $\sqsubseteq$ to refer to its order, which is dual to the natural one, $\leq$, but we usually prefer to work with the latter.

2.3. Intrinsic topologies

A subset $U \subseteq P$ of a poset P is *upper* if $x \sqsupseteq y \in U$ implies $x \in U$. Upper sets inaccessible by directed suprema form a topology called the *Scott topology*; it is denoted $\sigma(P)$ (or σ for short). A function $f \colon P \to Q$ between posets is Scott-continuous iff it preserves the order and suprema of directed subsets that exist in P. A continuous poset P admits a countable domain-theoretic basis iff its Scott topology is second countable. The collection $\{\mathord{\uparrow\!\!\uparrow} x \mid x \in P\}$ forms a basis for the Scott topology on a continuous poset P. The topology satisfies only weak separation axioms: It is always T_0 on a poset but T_1 only if the order is trivial. (A topological space (X, τ) is T_0 if and only if the relation R on X defined as xRy iff $\forall U \in \tau.\ (x \in U \Rightarrow y \in U)$ is a partial order (called the *specialisation order*). The space is T_1 if and only if its underlying specialisation order reduces to equality.) For an introduction to T_0 spaces, see [4]. An excellent general reference on Topology is [3].

2.4. Martin's theory

Now we will give a brief summary of the main elements of Keye Martin's theory of measurements. Our main reference is [5].

2.4.1. Quantitative approximation

Let P be a poset. For a monotone mapping $\mu\colon P \to Q$ between posets P and Q and any $x \in P$, $\varepsilon \in Q$ we define $\mu(x,\varepsilon) := \{y \in P \mid y \sqsubseteq x \wedge \varepsilon \ll \mu y\}$. One can readily see that in the case when $Q = [0,\infty)^{op}$ we can write $\mu(x,\varepsilon) = \{y \in P \mid y \sqsubseteq x \wedge \mu y < \mu x + \varepsilon\}$. We say that $\mu(x,\varepsilon)$ is the set of elements of P which are *ε-close to x*. The map μ can be thought of as a quantitative measure of a relative "distance" between elements in P.

2.4.2. Measurement

We say that a monotone mapping $\mu\colon P \to Q$ between posets P and Q *induces the Scott topology* on a subset X of P if

$$\forall U \in \sigma(P).\ \forall x \in X \cap U.\ \exists \varepsilon \in Q.\ x \in \mu(x,\varepsilon) \subseteq U.$$

We denote it as $\mu \to \sigma(X)$. Note that if $\mu\colon P \to [0,\infty)^{op}$, then the condition $\mu \to \sigma(P)$ reduces to

$$\forall U \in \sigma(P).\ \forall x \in U.\ \exists \varepsilon > 0.\ \mu(x,\varepsilon) \subseteq U.$$

Definition 1. *Let P be a continuous poset and let $\mu\colon P \to [0,\infty)^{op}$ be a Scott-continuous map. If $\mu \to \sigma(P)$, then we will say that μ* measures *P or that μ is a* measurement *on P. If $\mu \to \sigma(\mu^{-1}\{0\})$, then we will say that μ is a* kernel measurement *on P.*

We will often find it useful to employ the following characterization of measurements:

Proposition 1. **(Martin)**
A Scott-continuous mapping $\mu\colon P \to [0,\infty)^{op}$ on a continuous poset P is a measurement iff for all $x \in P$ and all sequences (x_n) in P with $(x_n) \ll x$ we have $\lim_{n\to\infty} \mu x_n = \mu x$ implies $\bigsqcup x_n = x$ and this supremum is directed.

Define the *kernel of μ* by $\ker\mu := \{x \in P \mid \mu x = 0\}$. The kernel is always a G_δ subset of maximal elements of P and as such is a topologically important object of study. We often seek a measurement on a domain with $\ker\mu = \max P$; this is called the *kernel condition* for measurements.

For the purpose of the next definition we introduce the following notation.

$$\mu(A,\varepsilon) := \bigcup\{\mu(x,\varepsilon) \mid x \in A\},$$

where A is a subset of a continuous poset P, the map $\mu\colon P \to [0,\infty)^{op}$ is monotone and $\varepsilon > 0$.

DEFINITION 2. *Let P be a continuous poset. A Scott-continuous map $\mu\colon P \to [0,\infty)^{op}$ is a* Lebesgue measurement *on P if for all Scott-compact subsets $K \subseteq \max P$ and for all Scott-open subsets $U \subseteq P$,*

$$K \subseteq U \cap \max P \quad \Rightarrow \quad \exists \varepsilon > 0.\ \mu(K,\varepsilon) \subseteq U \cap \max P.$$

3. Examples

3.1. The Cantor set model

Let Σ^∞ denote the set of all finite and infinite words over a finite alphabet Σ, with the prefix ordering. This is an ω-algebraic (ideal) domain. For all $x, y \in \Sigma^\infty$, $x \ll y$ holds iff $x \sqsubseteq y$ and x is finite. The mapping $2^{-|\cdot|}\colon \Sigma^\infty \to [0,\infty)^{op}$, where $|\cdot|\colon \Sigma^\infty \to \omega \cup \{\infty\}$ takes a string to its length is a Lebesgue measurement on Σ^∞ [5].

3.2. The powerset of naturals

The collection of all subsets of ω ordered by inclusion, $\mathcal{P}\omega$, is an ω-algebraic domain. The supremum of a directed set $S \subseteq \mathcal{P}\omega$ is $\bigcup S$ and for all elements x, y of $\mathcal{P}\omega$ the approximation relation is given by $x \ll y$ iff $x \subseteq y$ and x finite. The mapping $|\cdot|\colon \mathcal{P}\omega \to [0,\infty)^{op}$ given by $|x| := \sum_{n \notin x} 2^{-(n+1)}$ is a Lebesgue measurement on $\mathcal{P}\omega$ [5].

3.3. The interval domain

The collection of compact, connected intervals of the real line ordered under reverse inclusion, $\mathbf{I}\mathbb{R}$, is an ω-continuous dcpo. The supremum of a directed set $S \subseteq \mathbf{I}\mathbb{R}$ is $\bigcap S$ and for all intervals $x, y \in \mathbf{I}\mathbb{R}$ we have $x \ll y$ iff

x is contained in the interior of y. The length function $|\cdot|\colon \mathbf{I}\mathbb{R} \to [0,\infty)^{op}$ given by $|x| = b - a$, where $x = [a, b] \in \mathbf{I}\mathbb{R}$, is a Lebesgue measurement on $\mathbf{I}\mathbb{R}$ [5].

3.4. The formal ball model

Introduced in [2]. Let (X, d_X) be a metric space.

$$\mathbf{B}X := \{(x, r) \mid x \in X, r \in [0,\infty)\}$$

is a continuous poset ordered by $(x, r) \sqsubseteq (y, s)$ iff $d_X(x, y) \leq r - s$. The way-below relation is characterised by $(x, r) \ll (y, s)$ iff $d_X(x, y) < r - s$. One can show that $\mathbf{B}X$ is a dcpo iff the metric d_X is complete. The mapping $\mu\colon \mathbf{B}X \to [0,\infty)^{op}$ given by $\mu(x, r) = r$ is a Lebesgue measurement on $\mathbf{B}X$ [5].

4. Modelling topological spaces

We have seen that measurements are functions that represent quantitative approximation on domains. In this section we adopt a certain view on continuous posets and see them as *models of topological spaces*. The slogan is:

> *Domains are computational models of their spaces of maximal elements!*

Adopting this perspective is rich in consequences: we are often able to match structural properties of the modelling poset to properties of the modelled space. It was discovered by Mike Reed and Keye Martin (and announced at the First Irish Conference on the Mathematical Foundations of Computer Science and Information Technology two years ago) that existence of a measurement on the model corresponds to existence of a development for the modelled topological space. Here, we present an elementary proof of this fact (Theorem 1) that has been found independently, and show how one can use this result to give a new proof to one of the first metrization theorems in Topology (Corollary 1).

Definition 3. *A* model *of a topological space X is a continuous poset P together with a homeomorphism $\phi\colon X \to \max P$, where $\max P$ carries its subspace Scott topology inherited from P. A model P is* complete *if it is a dcpo;* countably based *if it is ω-continuous;* ideal *if it is an ideal poset.*

Below we present a general construction of a model for a wide class of first-countable topological spaces. Recall that a sequence $\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3,\ldots$ of open covers of a topological space (X,τ) is called a *development* for X provided that if $x \in U \in \tau$, then there exists $n \in \omega$ such that $x \in \mathbf{St}(x,\mathcal{U}_n) \subseteq U$, where $\mathbf{St}(x,\mathcal{U}_n) := \bigcup\{V \in \mathcal{U}_n \mid x \in V\}$. A topological space (X,τ) is *developable* if it admits a development.

Lemma 1. *If X is a developable and T_1 topological space, then it is homeomorphic to the subset of maximal elements of an ideal poset P equipped with a measurement $\mu\colon P \to [0,1)^{op}$ with $\ker\mu = \max P$.*

Proof. ($\Rightarrow$). Let $\{\mathcal{U}_n\}_{n\in\omega}$ be a development for X. We can assume that elements of each cover are nonempty. Define $X' := \{\{a\} \mid a \in X\}$ and $P := \bigcup \mathcal{U}_n \cup X'$. Assign a number $n(x) := \sup\{m \mid x \in \mathcal{U}_m\}$ to every element x in $P \setminus X'$ and put $n(x) := \infty$, whenever $x \in X'$.

Note that if $z \notin X'$, then $n(z) \in \omega$. For, suppose the contrary: $z \in \mathcal{U}_m$ for every $m \in \omega$. Since $z \notin X'$, there exist at least two distinct elements $a, b \in z$. By T_0 property of the space, there exists $v \in \tau$ with either $a \in v$ and $b \notin v$ or we have $a \notin v$ and $b \in v$. In the former case there is $k \in \omega$ such that $a \in \mathbf{St}(a,\mathcal{U}_k) \subseteq v$. But by our supposition, $z \subseteq \mathbf{St}(a,\mathcal{U}_k)$ and so $z \subseteq v$, a contradiction witnessed by $b \in z \setminus v$. The latter case is symmetric.

Consider a partial order $\sqsubseteq$ between elements of P defined as the reflexive closure of the following strict order: $x \sqsubset y$ iff $y \subseteq x$ and $n(y) > n(x)$. Note that if $x \in X'$ (say, $x = \{a\}$ for some $a \in X$) and $x \sqsubseteq y$ for some $y \in P$, then $y \subseteq \{a\}$ and hence $y = \{a\} = x$. Therefore, $X' \subseteq \max P$. Conversely, if $x \notin X'$, then $x \in \mathcal{U}_n$ for some $n \in \omega$. Choose any $b \in x$ and note that $x \sqsubset \{b\}$ in P. Hence, $x \notin \max P$. We conclude that $X' = \max P$.

Observe that

$$\forall x, y \in P.\ n(x) = n(y) \Rightarrow (x = y \vee x\#y), \tag{1}$$

by the definition of the order on P (recall that $x\#y$ means that the subset $\{x, y\}$ of P has no upper bound).

Define a mapping $\mu\colon P \to [0,1)^{op}$ by $\mu x = 2^{-n(x)}$ if $x \in P \setminus X'$ and $\mu x = 0$, otherwise. By definition and (1), $\ker\mu = X' = \max P$. It is also clear that the map is monotone and strictly monotone.

We will show that the function μ is Scott-continuous. Let D be a directed subset of P with supremum x.

Assume $x \in P \setminus X'$. Suppose that for any $d \in D$ we have $n(d) < n(x)$. Since D is nonempty, choose $d_1 \in D$ such that for any other $e \in D$, $n(e) \leq n(d_1)$. Now, if for arbitrary $d_2 \in D$ we have $d_2 \sqsubseteq d_1$, then $d_1 = x$, a contradiction as $n(d_1) < n(x)$. Otherwise, there is $d_2 \# d_1$ and by directness of D, there exists $d_3 \in D$ with $n(d_3) > n(d_1)$, a contradiction with our choice of d_1. We conclude that there exists an element $d \in D$ with $n(d) = n(x)$ and hence $x = d \in D$ by (1). We have proved that

$$\forall D \in P.\ (x = \bigsqcup^{\uparrow} D \wedge x \notin X') \Rightarrow x \in D. \tag{2}$$

Therefore, $\bigsqcup^{\uparrow} \mu(D) = \mu x$.

Assume that $x \in X'$. Suppose that there exists $m \in \omega$ such that $n(d) \leq m$ for any $d \in D$. Without loss of generality we may choose the number m in such a way that $m = n(e)$ for some $e \in D$. If all elements of D are below e, then $x \sqsubseteq e$ and hence $x = e$, by maximality of x. This implies that $n(e) = n(x) = 0$, a contradiction. Otherwise, there exists $e_1 \in D$ with $e_1 \# e$. By directness of D, there is $e_2 \sqsupseteq e_1, e$ with $n(e_2) > n(e)$, which is again a contradiction. We have shown that

$$\forall D \in P.\ (x = \bigsqcup^{\uparrow} D \wedge x \in X') \Rightarrow \{n(d) \mid d \in D\} \text{ is unbounded.} \tag{3}$$

Hence, $\bigsqcup^{\uparrow} \mu(D) = 0 = \mu x$.

We conclude that the mapping μ is Scott-continuous.

We claim that every non-maximal element is compact. Let $z \in P \setminus X'$ and $z \sqsubseteq x = \bigsqcup^{\uparrow} D$ for some directed subset D of P. If $x \notin X'$, then $z \sqsubseteq x \in D$ by (2). Otherwise, say $x = \{a\}$ for some $a \in X$, and so there exists $k \in \omega$ such that $a \in \mathbf{St}(a, \mathcal{U}_k) \subseteq z$. Without loss of generality, $k > n(z)$ and $n(e) = k$ for some $e \in D$ (the latter follows from (3)). Hence, $a \in e \subseteq z$ and so $z \sqsubseteq e$. We have shown that $z \ll z$, whenever $z \in P \setminus X'$.

It is now easy to see that for any $x \notin X'$ we have $\mathop{\downarrow\!\!\downarrow} x = {\downarrow} x$ and so $x = \bigsqcup^{\uparrow} \mathop{\downarrow\!\!\downarrow} x$. Otherwise, if $x \in X'$ (say $x = \{a\}$), then by construction of P, $\mathop{\downarrow\!\!\downarrow} x$ is directed and $\{n(y) \mid y \ll x\}$ is unbounded. Clearly, if $\mathop{\downarrow\!\!\downarrow} x \sqsubseteq z$ for any other $z \in P$, then $n(z) = \infty$ and so $z \in X'$. Then $z = x$ by the T_1 axiom of the space X. We conclude that $x = \bigsqcup^{\uparrow} \mathop{\downarrow\!\!\downarrow} x$. Therefore, P is an ideal poset. Also, from the construction of P it is immediate that $\tau = \sigma(P)_{|X'}$.

Finally, we will show that the mapping μ measures P. Let $x \in P$ and $x \in \mathop{\uparrow\!\!\uparrow} z \in P$. If $x \notin X'$, taking $\varepsilon := \mu x/2$ proves the claim. Otherwise, $x = \{a\}$ for some $a \in X$ and the claim follows from the fact that there exist $k \in \omega$ with $a \in \mathbf{St}(a, \mathcal{U}_k) \subseteq z$. $\square$

Theorem 1. **(Reed & Martin)**
For a topological space X, the following are equivalent:

1. *X is developable and T_1,*
2. *X is the kernel of a measurement on an ideal poset,*
3. *X is the kernel of a measurement on a continuous poset.*

Proof. (1)$\Rightarrow$(2) follows from the lemma above. (2)$\Rightarrow$(3) is trivial. (3)$\Rightarrow$(1) is proved by Martin in [5], Proposition 5.3.5, page 137. $\Box$

A metrizability theorem by Alexandroff and Urysohn (cf. e.g. [7], Theorem 23.7, page 169) is now a corollary of the above characterization.

Corollary 1. **(Alexandroff & Urysohn, 1923)**
A T_0 space is metrizable iff it has a development $\{\mathcal{U}_n\}_{n\in\omega}$ such that whenever $y, z \in \mathcal{U}_n$ and $y \cap z \neq \emptyset$, then $y \cup z \subseteq w$ for some $w \in \mathcal{U}_{n-1}$.

Proof. ($\Rightarrow$). Take $\mathcal{U}_n := \{B(x, \frac{1}{4^n}) \mid x \in X\}$. ($\Leftarrow$). T_1 separation follows easily from the assumptions. Then, build P as in Lemma 1. Take any pair of consistent elements $x, y \in P$. Then $x, y \in P \setminus X$. Without loss of generality suppose that $x \in \mathcal{U}_k$, $y \in \mathcal{U}_n$ and $k > n$. Since $\mathcal{U}_k$ refines $\mathcal{U}_n$, there exists $z \sqsubseteq x$ with $z \in \mathcal{U}_n$. By assumption, we can choose $w \in \mathcal{U}_{n-1}$ with $w \sqsubseteq z, y$. Therefore, $w \sqsubseteq x, z$ and $\mu w = 2\max\{\mu x, \mu y\}$. By Theorem 5.4.2 of [5], the function μ is a Lebesgue measurement on X. Hence, X is metrizable by Theorem 5.4.3 of [5]. $\Box$

5. References

[1] Abramsky S., Jung A.; *Domain Theory*, in: S. Abramsky, D.M. Gabbay and T.S.E. Maibaum, (eds.), *Handbook of Logic in Computer Science*, Clarendon Press, Vol. 3, 1994, pp. 1–168.

[2] Edalat A., Heckmann R.; *A computational model for metric spaces*, Theoretical Computer Science, Vol. 193(1–2), 1998, pp. 53–73.

[3] Engelking R.; *General Topology*, Heldermann Verlag, 1989.

[4] Heckmann R.; *Power Domain Constructions (Potenzbereich-Konstruktionen)*, Ph.D. thesis, Universität des Saarlandes, December 1990.

[5] Martin K.; *A Foundation for Computation*, Ph.D. thesis, Department of Mathematics, Tulane University, New Orleans 2000.

[6] Waszkiewicz P.; *Quantitative Continuous Domains*, Ph.D. thesis, School of Computer Science, The University of Birmingham, May 2002.

[7] Willard S.; *General Topology*, Addison-Wesley, 1970.

Received October 3, 2002